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Manifestation of Acceleration During Transient Heat Conduction

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The damped wave conduction and relaxation equation is derived from the free electron theory. The relaxation time is a third of the collision time between the electron and the obstacle in a given material. Six different reasons are given to seek a generalized Fourier's law of heat conduction. The hyperbolic governing equation is solved for by four different methods for three different boundary conditions. The reports in the literature of a temperature overshoot are revisited. For a small slab, $a < \pi(\alpha\tau_r)^{1/2}$, the temperature was shown to exhibit subcritical damped oscillations. In the case of the semi-infinite medium reports in the literature about a wave discontinuity were revisited. A substitution variable that is symmetric in space and time, that is, $\eta = \tau^2 - X^2$, is proposed to transform the governing equation into a Bessel differential equation. Three regimes are recognized in the solution: inertial lagging zero transfer regimes, a rising regime, and a third falling regime. The manifestation of the relaxation time for the case of the periodic boundary condition is studied using the method of complex temperature. The solution is an overdamped system. The storage coefficient is defined and found to be a critical parameter in the analysis.

Nomenclature

a	=	half-width of the slab, m
erf	=	error function $\int \exp(-s^2)\mathrm{d}s$
f	=	function of time
g	=	function of the spatiotemporal variable, η
I_0	=	modified Bessel function of the first kind and zeroth order
J_0	=	Bessel function of the first kind and zeroth order
K_0	=	modified Bessel function of the second kind and zeroth order
k	=	thermal conductivity, W/m/K
q	=	heat flux, W/m^2
q^*	=	dimensionless heat flux, $q/(k\rho C_p)1/2(T_s - T_0)$
T	=	temperature, K
T_s	=	surface temperature, K
T_0	=	initial temperature, K
u	=	dimensionless temperature $(T - T_s)/(T_0 - T_s)$
V	=	function of time only
v_h	=	velocity of heat, m/s
X	=	dimensionless distance, $x(\alpha\tau_r)^{1/2}$
X_a	=	dimensionless half-width, $a(\alpha\tau_r)^{1/2}$
x	=	distance, m
Y_0	=	Bessel function of the second kind and zeroth order
α	=	thermal diffusivity, m^2/s
δ	=	Kronecker delta function
η	=	transformation variable, $\eta = X^2 - \tau^2$
λ_n	=	lambda function
τ	=	dimensionless time, t/τ_r
τ_r	=	relaxation time, s

I. Introduction

ONSAGER [1] argued that the Fourier's [2] law of heat conduction contradicts the principle of microscopic reversibility. Fourier's law is only an approximation of the description of transient conduction and neglects the time needed for the acceleration of heat flow. The time needed for acceleration may be small and could be of the same order of magnitude as that of the collision time between molecules. To remove the paradox in the Fourier conduction model assuming an infinite speed of propagation, the damped wave conduction and relaxation equation was originally suggested by Maxwell [3], Morse and Feshbach [4], and postulated independently by Cattaneo [5] and Vernotte [6]. Reviews on heat waves have been presented by Joseph and Preziosi [7,8] and Osizik and Tzou [9]. Very little work has been reported on the connection between macroscopic laws and miscroscale phenomena and the manifestation of the miscroscale phenomena in the macroscale. A comprehensive insight into the analytical solutions using the manifestations of the generalized transport equation is given by Sharma [10].

The Cattaneo and Vernotte equation was found to be admissible within the framework of the second law of thermodynamics (Tzou [11]). Bai and Lavine [12] discussed the damped wave equation in the context of the second law of thermodynamics. The relaxation time was suggested to be viewed as the time lag between the heat flux vector and the temperature gradient when the response time is short. Some investigators have related the relaxation time to the electron–phonon collisions and the volumetric heat capacities of the electrons and the metal lattice (Ozisik and Tzou [13]). Nernst [14] suggested that at low temperatures in good thermal conductors heat may have sufficient "inertia" to give rise to oscillatory discharge. Landau [15] found two speeds, one for ordinary sound and one for a second sound, which describe propagation waves of temperature. In Landau's theory there is no damping or dissipation and both sound speeds are associated with wave equations rather than telegraph equations. Propagation of Landau waves in Helium II is specifically a quantum phenomenon. Other forms of non-Fourier heat conduction are the ballistic transport equations due to Chen [16], dual phase lag model by Tzou [17], Jeffreys [18] and the equation of phonon radiative transport (EPRT), equation of phonon transport proposed by Majumdar [19], and the microscopic two-step model suggested by Qiu and Tien [20]. An infinite order partial differential equation (PDE) was presented by Sharma [21] using Taylor series expansion to fully describe the transient events. Experimentally measured values of relaxation times were reported on the order of a few microseconds by Tzou [17] for steel at 400°C. Mitra et al. [22] and Kaminski [23] have measured relaxation times for materials with a nonhomogeneous inner structure of the order of 15–20 s.

II. Limitations to Fourier's Law of Heat Conduction

The motivation for seeking a generalized law for heat conduction where Fourier's law becomes the particular case is sixfold. The second is some of the singularities found in the description of transient heat conduction using the parabolic conduction equations, with the first being the theory of Onsager. When transient

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temperature events are described using Fourier's law a "blowup" occurs during short contact times in the expression for surface flux. This is so for the cases of (Sharma [10]) 1) surface flux expression in a semi-infinite body subject to a step change in one of the boundary temperatures, 2) surface flux for a finite slab subject to constant wall temperature on either of its edges, 3) a temperature term in the constant wall flux problem in cylindrical coordinates in a semi-infinite medium solved by the Boltzmann transformation leading to a solution in exponential integral, and 4) in the short time limit, the solution from the parabolic conduction equations for a semi-infinite sphere using the similarity transformation. The third reason for seeking alternate forms to Fourier's law of heat conduction is because light is the speediest of velocities. On examining the solution for the transient temperature Landau and Lifshitz [24] noted that for times greater than zero the temperature is finite at all points in the infinite medium except at infinite location. It can be inferred that the heat pulse has traveled at infinite speed. This is in conflict with the light speed barrier stated by the theory of relativity of Einstein. The fact that any speed of a moving object, including the thermal wave, must be less than the speed of light was examined by Kelly [25] for diffusion. The fourth reason is the realization of the empirical nature of the development of the Fourier's law of heat conduction, from observations at steady state. Use of it in the transient state is an extrapolation. The fifth is the overprediction of theory to experiment in an important industrial process such as fluidized bed heat transfer to surfaces, chromatography, CPU overheating, adsorption, gel electrophoresis, restriction mapping, laser heating of semiconductors during manufacture of semiconductor devices and drug delivery systems. A sixth reason is that Fourier's law breaks down also at small scales (Bejan [26], Casimir [27]). In this limit the flux is described by an expression similar to the one used in radiation heat transfer (Swartz and Pohl [28]). The heat transport, for example, in dielectric crystalline materials is believed to be primarily by atomic or crystal vibrations. These vibrations travel as waves and the energy of the waves quantitated is the phonon (Kittel [29]).

An attempt is made to improve the solutions presented for the semi-infinite medium by Baumeister and Hamill [30] by developing a transformation, $\eta = \tau^2 - X^2$. This is seen to transform the hyperbolic PDE in two variables of space and time after removal of the damping term to a Bessel differential equation. Readily usable solutions can be obtained by realizing that the integration constant from the Bessel solution is with respect to the transformation variable which is a function of two variables. In the case of ordinary differential equations, the integration constants need to be solved for. In partial differential equations these can be a variable of a single function. This way the discontinuity seen in the Baumeister and Hamill solution is removed and the general solution improved upon. The finite slab problem is revisited and bounded solutions obtained by two different methods. In the first method the separation of variables is used and the final condition in time for the wave temperature is used to obtain a bounded Fourier series solution. In this approach the hyperbolic PDE is first divided by an $\exp(-\tau/2)$ and then the resulting equation in wave temperature was solved. In a second method the damped wave conduction and relaxation equation is directly solved without dividing the equation with the $\exp(-n\tau)$ where $n = \frac{1}{2}$, by a hybridized method of relativistic transformation of coordinates and the method of separation of variables. A hybrid method of substitution and separation of variables is used in this paper to obtain an exact solution to the hyperbolic heat equation for a finite slab. The solution to the equation is shown to be within the bounds of the second law of thermodynamics. The solution from both methods is well bounded and does not overshoot as indicated by Taitel [31]. The periodic boundary condition is also studied using the method of complex temperature.

III. Derivation of Damped Wave Conduction and Relaxation Equation from Free Electron Theory

Further in the literature very little attention is paid to the derivation of the Cattaneo and Vernotte equation. Ali [32,33] has used statistical mechanics and the kinetic theory and attempted to derive Cattaneo's heat equation for monatomic and diatomic gases. Glass and McRae [34] looked at the variable specific heat and thermal relaxation parameter. In this study the damped wave conduction and relaxation equation is derived from the free electron theory. The derivation of Ohm's law of electric conduction is revisited to obtain the damped wave momentum transfer and relaxation equation by analogy. The electrical resistivity of materials differs by 30 orders of magnitude, so that a single theory to explain the behavior of all materials may be difficult to develop. In the free electron model the outermost electrons of the atoms can take part in conduction. They are not bound to the atom but are free to move through the whole solid. These electrons have been variously called the free electron cloud, the free electron gas, or the Fermi gas. The assumption is that the potential field due to the ion cores is uniform throughout the solid. The free electrons have the same potential energy everywhere in the solid. Because of the electrostatic attraction between a free electron and the ion core this potential energy will be a finite negative value. Only energy differences are important and the constant potential can be taken to be zero. Then the only energy that has to be considered is the kinetic energy. The kinetic energy is substantially lower than that of the bound electrons in an isolated atom as the field of motion for the free electron is considerably enlarged in the solid as compared to the field around an isolated atom. The free electron theory can be used to better understand electrical conduction. By Lorenz analogy the heat conduction can also be predicted in a similar manner. The independent electron assumption was developed by Drude in 1905. Some of the assumptions in the free electron theory claim that electrons are responsible for all of the conduction. The electrons behave like an ideal gas, occupy negligible volume, undergo collisions, and are perfectly elastic. Electrons are free to move in a constrained flat bottom well. Electron distribution of energy is a continuum.

The general equation of motion for the drift velocity of the free electron on account of an applied temperature gradient driving force can be given by the following expression from the Drude theory:

$$m \cdot \mathrm{d}v_e/\mathrm{d}t + mv_e/\tau = -3/2k_B \mathrm{d}T/\mathrm{d}x \tag{1}$$

where m is the mass of the electron, v_e the drift velocity, τ is the collision time of the electron with an obstacle, k_B the Boltzmann constant, and $\mathrm{d}T/\mathrm{d}x$ the applied temperature gradient. The drift velocity of the electron is different from the random velocities associated with it. It is superimposed on the random motion. It is in a net direction of the superimposed field. This leads to a net flow of charge and the passage of electric current. The electron encounters obstacles during drift and the directional motion is lost and reduced to the random motion. The memory gained is lost and the clock is set back to zero. Collisions occur in the time interval t. The rate of destruction of momentum by virtue of the collision is given by mv_e/τ. This slows down or drags down the electron. The drag force will balance the applied force due to the temperature gradient at steady state to yield the Fourier's law of heat conduction. This can be seen in the following steps:

$$v_e = -3\tau/2mk_B \mathrm{d}T/\mathrm{d}x \tag{2}$$

The heat flux can be defined as

$$q = n(3/2k_BT)v_e \tag{3}$$

where n is the number of electrons per unit volume where $(3/2k_BT)$ is the average energy of the electron from the equipartition energy theorem. Using the Boltzmann relation the heat flux can also be written as

$$q = n\tfrac{1}{2}mv_e^3 \tag{4}$$

Multiplying Eq. (2) throughout by $n(3/2k_BT)$ and using Eq. (3), Eq. (2) becomes

$$q = -9nT\tau/4mk_B^2 \mathrm{d}T/\mathrm{d}x = -k\partial T/\partial x \tag{5}$$

where the thermal conductivity can be written as

$$k = (9nT\tau/4mk_B^2) \quad (6)$$

During transient heat conduction the acceleration term may become important. Rewriting Eq. (1) as

$$\tau \mathrm{d}v_e/\mathrm{d}t + v_e = -3\tau/2mk_B \mathrm{d}T/\mathrm{d}x \quad (7)$$

Multiplying Eq. (7) throughout by $n(3/2k_BT)$ and combining with Eqs. (3) and (6):

$$n(3/2k_BT)\tau \mathrm{d}v_e/\mathrm{d}t + q = -k\partial T/\partial x \quad (8)$$

Using the Boltzmann relation $(1/2mv_e^2 = 3/2k_BT)$, Eq. (8) becomes

$$\tfrac{1}{2}n\tau m v_e^2 \mathrm{d}v/\mathrm{d}t + q = -k\partial T/\partial x \quad (9)$$

Differentiating Eq. (4) with respect to t:

$$\partial q/\partial t = 3/2nmv_e^2 \mathrm{d}v_e/\mathrm{d}t \quad (10)$$

Combining Eq. (10) with Eq. (9):

$$t/3\partial q/\partial t + q = -k\partial T/\partial x \quad (11)$$

Equation (11) is equivalent to the Cattaneo and Vernotte equation given by Eq. (12) when $\tau/3 = \tau_r$:

$$\tau_r \partial q/\partial t + q = -k\partial T/\partial x \quad (12)$$

IV. Transient Heat Conduction and Relaxation in a Finite Slab Subject to Constant Wall Temperature

A. Taitel Paradox and the Final Time Condition

Previous reports, Taitel [31], Barletta and Zanchini [35], have raised some concerns about the second law of thermodynamics and the Cattaneo and Vernotte equation. Taitel considered heat conduction in an infinitely wide parallel slab with thickness 2L such that the thermal conductivity, the thermal diffusivity, the specific heat at constant volume, and the thermal relaxation time of the slab can be considered constant. They note that at times zero, $\partial T/\partial t = 0$ and use it as one of the time conditions and $T = T_0$ at times zero as the second time condition. For times greater than zero, the temperature distribution on the two sides of the slab is kept uniform with a value $T_w \neq T_0$. By symmetry at the center of the slab, $\partial T/\partial x = 0$, is the fourth space condition. A second order hyperbolic PDE can be completely described by two space and two time conditions. Upon obtaining the transient temperature Taitel points out that the absolute value of the temperature change $(T - T_0)$ may exceed $|T_w - T_0|$. Barletta and Zanchini [35] develop a solution for the finite slab problem by the method of separation of variables. They show by a plot of $1 - u$ vs X for Vernotte number 1 $(\alpha\tau_r/4L^2)$ and Fourier number of 0.7 $(\alpha t/4L^2)$, that $|T - T_0|$ may exceed $|T_w - T_0|$ as pointed out by Taitel. In another plot of $1 - u$ vs X for Vernotte number 1 and Fourier number of 0.25, the equilibrium value for the temperature was attained by an oscillatory process. The parabolic conduction predicts a continuous increase in temperature from zero to one at any internal position. The solution obtained by Taitel [31] for the centerline temperature of the finite slab is given below. They considered a constant wall temperature and the initial time conditions included a $\partial T/\partial t = 0$ term in addition to the initial temperature condition. The exact solution presented by Taitel is as follows:

$$u = \sum_0^\infty b_n \exp(-\tau/2)\exp\Big(-\tau/2\sqrt{[1 - 4(2n+1)^2\pi^2\alpha\tau_r/a^2]} + \sum_0^\infty c_n \exp(-\tau/2)\exp\Big\{+\tau/2\sqrt{[1 - 4(2n+1)^2\pi^2\alpha\tau_r/a^2]}\Big\}\Big) \quad (13)$$

Multiplying both sides of the equation by $\exp(\tau/2)$,

$$u\exp(\tau/2) = W = \sum_0^\infty b_n \exp\big(-\tau/2\sqrt{[1 - 4(2n+1)^2\pi^2\alpha\tau_r/a^2]} + \sum_0^\infty c_n \exp\{+\tau/2\sqrt{[1 - 4(2n+1)^2\pi^2\alpha\tau_r/a^2]}\}\big) \quad (14)$$

At infinite times, the LHS of Eq. (19) is 0 times ∞ and is zero. The RHS does not vanish. Thus the expression given by Taitel [31] and later discussed as a temperature overshoot may be a result of the growing exponential term in the above expression. Sharma [10] considered a finite slab of width $2a$ with an initial temperature at T_0. The sides of the slab are maintained at a constant temperature of T_s. The governing equation in the dimensionless form is then

$$\partial u/\partial\tau + \partial^2 u/\partial\tau^2 = \partial^2 u/\partial X^2 \quad (15)$$

where

$$u = (T - T_s)/(T_0 - T_s); \qquad \tau = \tau/\tau_r; \qquad X = x/\sqrt{(\alpha\tau_r)} \quad (16)$$

The initial condition is given as follows:

$$t = 0, \qquad Vx, \qquad T = T_0, \qquad u = 1 \quad (17)$$

Boundary conditions in space

$$t > 0, \qquad x = 0, \qquad \partial T/\partial x = 0, \qquad \partial u/\partial x = 0 \quad (18)$$

$$t > 0, \qquad x = \pm a, \qquad T = T_s, \qquad u = 0 \quad (19)$$

The fourth and final condition in time is

$$t = \infty, \qquad Vx, \qquad T = T_s, \qquad u = 0 \quad (20)$$

The governing equation was obtained by a one dimensional energy balance (in $-$ out $+$ reaction $=$ accumulation). This is achieved by eliminating q_x between the damped wave conduction and relaxation equation and the equation from energy balance $[-\partial q/\partial x = (\rho C_p)\partial T/\partial t]$. This is achieved by differentiating the constitutive equation with respect to x and the energy equation with respect to t and eliminating the second cross derivative of q with respect to x and time. This equation is then nondimensionalized. The solution is obtained by the method of separation of variables. Let

$$u = V(\tau)\phi(X) \quad (21)$$

Equation (31) becomes

$$\phi''(X)/\phi(X) = [V'(\tau) + V''(\tau)]/V(\tau) = -\lambda_n^2 \quad (22)$$

$$\phi(X) = c_1\sin(\lambda_n X) + c_2\cos(\lambda_n X) \quad (23)$$

From the boundary conditions, at

$$X = 0, \qquad \partial\phi/\partial X = 0, \qquad c_1 = 0 \quad (24)$$

$$\phi(X) = c_1\cos(\lambda_n X) \quad (25)$$

$$0 = c_1\cos(\lambda_n X_a) \quad (26)$$

$$(2n - 1)\pi/2 = \lambda_n X_a \quad (27)$$

$$\lambda_n = (2n - 1)\pi\sqrt{(\alpha\tau_r)}/2a, \qquad n = 1, 2, 3, \ldots \quad (28)$$

The time domain solution would be

$$V = \exp(-\tau/2)\Big\{c_3\exp\Big[\sqrt{(1/4 - \lambda_n^2)}\tau\Big] + c_4\exp\Big[-\sqrt{(1/4 - \lambda_n^2)}\tau\Big]\Big\} \quad (29)$$

or

$$V\exp(\tau/2) = \left\{c_3 \exp\left[\sqrt{(1/4-\lambda_n^2)}\tau\right] + c_4 \exp\left[-\sqrt{(1/4-\lambda_n^2)}\tau\right]\right\} \quad (30)$$

From the final condition $u = 0$ at infinite time. $V\phi\exp(\tau/2) = W$ is the wave temperature at infinite time. The wave temperature is that portion of the solution that remains after dividing the damping component either from the solution or the governing equation. For any nonzero ϕ, it can be seen that at infinite time the LHS of Eq. (30) is a product of zero and infinity and a function of x and is zero. Hence the RHS of Eq. (30) is also zero and hence in Eq. (30) c_3 needs to be set to zero. Therefore,

$$u = \sum_1^\infty c_n \exp(-\tau/2)\exp\left[-\sqrt{(1/4-\lambda_n^2)}\tau\right]\cos(\lambda_n X) \quad (31)$$

where λ_n is described by Eq. (28). C_n can be shown using the orthogonality property to be $4(-1)^{n+1}/(2n-1)\pi$. It can be seen that Eq. (31) is bifurcated. As the value of the thickness of the slab changes the characteristic nature of the solution changes from monotonic exponential decay to subcritical damped oscillatory. For $a < \pi\sqrt{(\alpha\tau_r)}$, even for $n = 1$, $\lambda_n > 1/2$. This is when the argument within the square root sign in the exponentiated time domain expression becomes negative and the result becomes imaginary. Using Demovrie's theorem and taking a real part for the small width of the slab,

$$u = \sum_1^\infty c_n \exp(-\tau/2)\cos\left[\sqrt{(\lambda_n^2 - 1/4)}\tau\right]\cos(\lambda_n X) \quad (32)$$

Equations (31) and (32) can be seen to be well bounded. Equation (32) becomes zero after some time. This would be time taken to reach steady state. Thus, for $a \geq \pi\sqrt{(\alpha\tau_r)}$

$$u = \sum_1^\infty c_n \exp(-\tau/2)\exp\left[-\sqrt{(1/4-\lambda_n^2)}\tau\right]\cos(\lambda_n X) \quad (33)$$

where $c_n = 4(-1)^{n+1}/(2n-1)\pi$ and $\lambda_n = (2n-1)\pi\sqrt{(\alpha\tau_r)}/2a$

The centerline temperature for a particular example is shown in Fig. 1. Eight terms in the infinite series given in Eq. (31) were taken and the values calculated on a 1.9 GHz Pentium IV desktop personal computer. The number of terms was decided on the incremental change or improvement obtained by doubling the number of terms. The number of terms was arrived at a 4% change in the dimensionless temperature. The results for the case of a small slab are shown in Fig. 2. The subcritical damped oscillations can be seen. The time taken to steady state can be read from the x intercept. In Fig. 3 a parametric study of the relaxation time is shown. A small slab of thickness of 1 cm and thermal diffusivity of 10^{-5} m^2/s is considered. Twelve terms were taken in the infinite series solution and four different relaxation times were calculated. The accuracy of the data was less than 4%. For the case when the relaxation time was small, that is, when Eq. (33) was applicable for the solution the centerline temperature decayed monotonically with the x axis as its asymptote. This happens when

$$\tau_r > (a^2/\pi^2\alpha) \quad (34)$$

When the relaxation time considered is large, in such a fashion that Eq. (32) is applicable the subcritical damped oscillations can be seen. The time taken to steady state can be read from the x intercept in such cases. At infinite relaxation time the governing equation will revert to the wave equation [10] and the d'Alembert solution will result. For a wide range of thermal relaxation times this approach can be seen to be viable.

The Taitel paradox is obviated by examining the final steady state condition and expressing the state in mathematical terms. The W term which is the dimensionless temperature upon removal of the damping term needs to go to zero at infinite time. This resulted in our solution being different from previous reports (Taitel [31], Barletta and Zanchini [35]) and is well bounded. The use of the final condition is may be what is needed for this problem to be used extensively in engineering analysis without being branded as violating the second law of thermodynamics. The conditions that are the touted violations of the second law are not physically realistic. A bifurcated solution results. For a small width of the slab, $a < \pi\sqrt{(\alpha\tau_r)}$, the transient temperature is subcritical damped oscillatory. The centerline temperature is shown in Fig. 2.

An exact well-bounded solution that is bifurcated depending on the width of the slab is provided. The transient solution to the damped wave Cattaneo and Vernotte non-Fourier hyperbolic wave

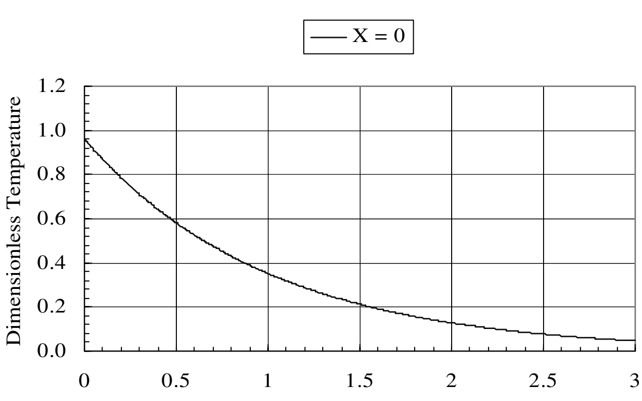

Fig. 1 Centerline temperature in a finite slab at constant wall temperature (large a) ($a = 0.86$ m, $\alpha = 10^{-5}$ m^2/s, and $\tau_r = 15$ s).

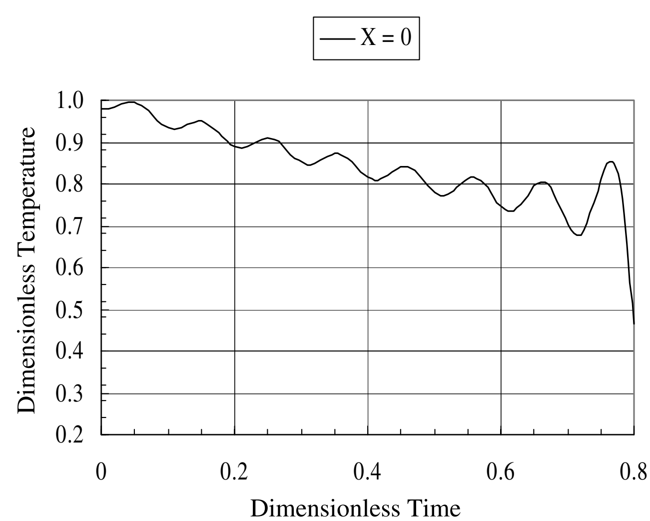

Fig. 2 Centerline temperature in a finite slab at constant wall temperature (small a) ($a = 0.001$ m, $\alpha = 10^{-5}$ m^2/s, and $\tau_r = 15$ s).

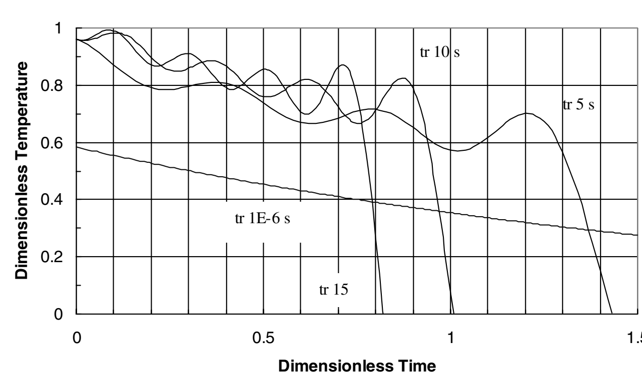

Fig. 3 Centerline temperature of a finite slab at different relaxation times thermal diffusivity = 1e-5 m^2/s; $a = 0.01$ m; a) $\tau_r = 1e\text{-}6$ s; b) $\tau_r = 5$ s; c) $\tau_r = 10$ s; d) $\tau_r = 15$ s.

propagative and relaxation equation is obtained by the method of separation of variables. A well-bounded infinite series expression is provided. The temperature overshoot identified by Taitel [31] is obviated by examining the steady state condition and expressing the state in mathematical terms. A bifurcated solution results. For a small width of the slab, $a < \pi\sqrt{(\alpha\tau_r)}$, the transient temperature is subcritical damped oscillatory. In both (Taitel [31] and Barletta and Zanchini [35]), there were four conditions used for initial and boundary constraints. The two in space domain are retained here. The initial temperature at time zero is also retained. However the slope with the time domain of the temperature at time zero is replaced with the final condition for the time domain, that is, at steady state the transient temperature will decay out to a constant value or to zero in the dimensionless form. This consideration is shown to change the nature of the solution considerably to a well-bounded expression that is bifurcated. For small values of the slab the transient temperature is subcritical damped oscillatory. For other values the Fourier series representation is augmented by a modification to the exponential time domain portion of the solution. In this section, the use of the final condition at steady state as the fourth condition to give a bounded solution in obeyance of Clausius inequality was achieved.

B. ALITER: Hybridized Method of Transformation and Separation of Variables

Consider a finite slab of width $2a$. The initial temperature is held at T_0 at zero time. For times greater than zero, $t > 0$, either surface of the finite slab was subject to a step change in temperature to T_s. Equation (15) can be solved by a hybridized method of transformation and separation of variables. Let $\eta = X^2 - \tau^2$. Then, Eq. (15) can be written as

$$\partial u/\partial\tau = 4\eta\partial^2 u/\partial\eta^2 + 4\partial u/\partial\eta \tag{35}$$

Equation (35) is now a PDE in two variables, that is, time and the transformation variable. This can be solved for the method of separation of variables. Let $u = g(\eta)\phi(\tau)$. Equation (35) then becomes

$$\phi'/\phi = (4\eta g'' + 4g')/g = -f \tag{36}$$

The LHS of Eq. (36) will be equal to the expression in the transformed variable even when the RHS of Eq. (51) is a function of time only. By letting f be a function of time only the solution to the equation describing the transformation variable can be seen to be a Bessel equation. Thus,

$$\eta^2 g'' + \eta g' + \eta f g/4 = 0 \tag{37}$$

As the transformation variable is a function of time and space the constants in the Bessel differential equation can be functions of one variable. Comparing Eq. (37) with the generalized Bessel equation, $a = 1$, $b = 0$, $c = 0$, $d = f/4$, $s = 1/2$, $p = 0$, and $\sqrt{|d|/s}$. Thus,

$$g = c_1 J_0[f^{1/2}(X^2 - \tau^2)^{1/2}] + c_2 Y_0[f^{1/2}(X^2 - \tau^2)^{1/2}] \tag{38}$$

At the wave front the temperature is finite and hence c_2 needs to be set to zero as Y_0 becomes infinitely large at the wave front. At the edge of the slab $u = 0$. Thus,

$$f^{1/2}(X_a^2 - \tau^2)^{1/2} = 2.4048 + (n-1)\pi \tag{39}$$

where 2.4048 is the first root of the Bessel function of the first kind and zeroth order. The subsequent roots of the Bessel function differ by nearly π. Thus,

$$f_n^{1/2} = [2.4048 + (n-1)\pi]/(X_a^2 - \tau^2)^{1/2} \tag{40}$$

Thus,

$$g = c_1 J_0[f^{1/2}(X^2 - \tau^2)^{1/2}/(X_a^2 - \tau^2)^{1/2}] \tag{41}$$

$$\phi'/\phi = -[2.4048 + (n-1)\pi]^2/(X_a^2 - \tau^2) \tag{42}$$

$$-\ln(\phi) = [2.4048 + (n-1)\pi]^2/(X_a^2 - \tau^2) \tag{43}$$

Integrating both sides and using the method of partial fractions,

$$\phi = \{(X_a + \tau)/(X_a - \tau)\}_A^{-1/2X[2.4048+(n-1)\pi]^2} \tag{44}$$

The general solution for the temperature for $X_a > \tau$ is then

$$u = \sum_1^\infty c_n \phi J_0\{2.4048 + (n-1)\pi[(X^2 - \tau^2)/(X_a^2 - \tau^2)]^{1/2}\} \tag{45}$$

c_n can be solved by using the principle of orthogonality for Bessel functions using the initial condition. At time zero ϕ in Eq. (45) that is given by Eq. (44) becomes 1. Multiplying both sides by $J_0(\lambda_m x/a)$ where $\lambda_m = 2.4048 + (n-1)\pi$ and integrating between $x = 0$ and $x = a$ all the terms in the infinite series except when $m = n$ integrate to zero. Thus,

$$c_n = \int_0^a J_0(\lambda_m x/a) \bigg/ \int_0^a J_0^2(\lambda_m x/a)\mathrm{d}x \tag{46}$$

C. Relativistic Transformation of Coordinates in Semi-Infinite Medium at Constant Wall Temperature

The semi-infinite medium is considered to study the spatio-temporal patterns that the solution of the non-Fourier damped wave conduction and relaxation equation exhibits. This kind of consideration has been used in the study of Fourier heat conduction. The boundary conditions can be different kinds such as the constant wall temperature or the constant wall flux (CWF), pulse injection, convective, insulated and exponential decay. The similarity or Boltzmann transformation worked out well in the case of parabolic PDE as shown in the previous section. The conditions at infinite width and zero time are the same. The conditions at zero distance from the surface and at infinite time are the same.

Baumeister and Hamill [30] solved the hyperbolic heat conduction equation in a semi-infinite medium subjected to a step change in temperature at one of its ends using the method of Laplace transform. The space integrated expression for the temperature in the Laplace domain had the inversion readily available within the tables. This expression was differentiated using Leibniz's rule and the resulting temperature distribution was given for $\tau > X$ as

$$u = (T - T_0)/(T_s - T_0) = \exp(-X/2) + X\int_X^\tau \exp(-p/2) I_1(p^2 - X^2)^{1/2}/(p^2 - X^2)^{1/2}\mathrm{d}p \tag{47}$$

The method of relativistic transformation of coordinates is evaluated to obtain the exact solution for the transient temperature. Consider a semi-infinite slab at initial temperature T_0, imposed by a constant wall temperature T_s for times greater than zero at one of the ends. The transient temperature as a function of time and space in one dimension is obtained. Obtaining the dimensionless variables

$$u = (T - T_0)/(T_s - T_0), \qquad \tau = t/\tau_r$$
$$X = x/\sqrt{(\alpha\tau_r)}, \qquad q^* = q/(k\rho C_p/\tau_r)^{1/2}/(T_s - T_0) \tag{48}$$

The energy balance on a thin spherical shell at x with thickness Δx is written in one dimension as $-\partial q/\partial x = \rho C_p \partial T/\partial t$. The governing equation can be obtained in terms of the heat flux after eliminating the temperature between the energy balance equation and the non-Fourier expression. This is achieved by differentiating Eq. (12) with respect to time and the energy balance equation with respect to x and then eliminating the second cross derivative of the temperature with respect to space and time:

$$\partial q^*/\partial\tau + \partial q^*/\partial\tau^2 = \partial q^* \partial X^2 \tag{49}$$

It can be seen that the governing equation for the dimensionless heat flux is identical in form with that of the dimensionless temperature. The initial condition is

$$\tau = 0, \qquad q^* = 0 \tag{50}$$

The boundary conditions are

$$X = \infty, \qquad q^* = 0 \tag{51}$$

$$X = 0, \qquad T = T_s, \qquad u = 1 \tag{52}$$

Let us suppose that the solution for q^* is of the form $W\exp(-nt)$ for $t > 0$ where W is the transient wave flux. Then

$$\exp(-n\tau)W(-n+n^2) + \exp(-n\tau)\partial W/\partial t(1-2n) + \exp(-n\tau)\partial^2 W/\partial\tau^2 = \exp(-n\tau)(\partial^2 W/\partial X^2) \tag{53}$$

For $n = 1/2$ Eq. (72) becomes

$$\partial W/\partial\tau^2 - W/4 = \partial^2 W/\partial X^2 \tag{54}$$

The solution to Eq. (54) can be obtained by the following relativistic transformation of coordinates, for $\tau > X$. Let $\eta = (\tau^2 - X^2)$. Then Eq. (54) becomes

$$\partial^2 W/\partial t^2 = 4t^2\partial^2 W/\partial\eta^2 + 2\partial W/\partial\eta \tag{55}$$

$$\partial^2 W/\partial X^2 = 4X^2\partial^2 W/\partial\eta^2 - 2\partial W/\partial\eta \tag{56}$$

Combining Eqs. (55) and (56) into Eq. (54),

$$4(\tau^2 - X^2)\partial^2 W/\partial\eta^2 + 4\partial W/\partial\eta - W/4 = 0 \tag{57}$$

or

$$\eta^2\partial^2 W/\partial\eta^2 + \eta\partial W/\partial\eta - \eta W/16 = 0 \tag{58}$$

Equation (58) can be seen to be a special differential equation in one independent variable. The number of variables in the hyperbolic PDE has thus been reduced from two to one. Comparing Eq. (58) with the generalized form of Bessel's equation it can be seen that $a = 1$, $b = 0$, $c = 0$, $s = 1/2$, and $d = -1/16$. The order of the solution is calculated as 0 and the general solution is given by

$$W = c_1 I_0(1/2\eta^{1/2}) + c_2 K_0(1/2\eta^{1/2}) \tag{59}$$

The wave flux W is finite when $\eta = 0$ and hence it can be seen that c_2 can be seen to be zero. The c_1 can be solved from the boundary condition given in Eq. (52). The expression for the dimensionless heat flux for times τ, greater than X is thus

$$q^* = c_1\exp(-\tau/2)I_0[1/2(\tau^2 - X^2)^{1/2}] \tag{60}$$

For large times, the modified Bessel's function can be given as an exponential and reciprocal in square root of time by asymptotic expansion. Consider the surface flux, that is, when in Eq. (60) X is set as zero.

$$q^* = c_1\exp(-\tau/2)\exp(\tau/2)/\sqrt{(2\pi\tau)} = c_1/\sqrt{(2\pi\tau)} \tag{61}$$

For times when $\exp(\tau)$ is much greater than the heat flux it can be seen that the second derivative in time of the dimensionless flux in Eq. (49) can be neglected compared with the first derivative. The resulting expression is the familiar expression for surface flux from the Fourier parabolic governing equation for constant wall temperature in a semi-infinite medium and is given by

$$q^* = 1/\sqrt{(2\pi\tau)} \tag{62}$$

Comparing Eqs. (61) and (62) it can be seen that c_1 is 1. Thus the dimensionless heat flux is given by

$$q^* = \exp(-\tau/2)I_0[1/2(\tau^2 - X^2)^{1/2}] \tag{63}$$

The solution for q^* needs to be converted to the dimensionless temperature u and then the boundary condition applied. From the energy balance,

$$-\partial q^*/\partial X = \partial u/\partial t \tag{64}$$

thus differentiating Eq. (64) with respect to X and substituting in Eq. (63) and integrating both sides with respect to τ. For $t > X$,

$$u = \int X\exp(-\tau/2)I_1(1/2(\tau^2 - X^2)^{1/2})/(\tau^2 - X^2)^{1/2}\mathrm{d}\tau + C(X) \tag{65}$$

It can be left as an indefinite integral and the integration constant can be expected to be a function of space. The $C(X)$ can be solved by examining what happens at the wave front. At the wave front $\eta = 0$ and time elapsed equals the time taken for a thermal disturbance to reach the location x given the wave speed $\sqrt{(\alpha/\tau_r)}$. The governing equations for the dimensionless heat flux and dimensionless temperature are identical in form. At the wave front, Eq. (54) reduces to

$$\partial W/\partial\eta = W/16 \tag{66}$$

or

$$W = c'\exp(\eta/16) = c' \tag{67}$$

$$u = c'\exp(-\tau/2) = c'\exp(-X/2) \tag{68}$$

Thus $C(X) = c'\exp(-X/2)$. Thus

$$u = \int X\exp(-\tau/2)I_1(1/2(\tau^2 - X^2)^{1/2})/(\tau^2 - X^2)^{1/2}\mathrm{d}\tau + c'\exp(-X/2) \tag{69}$$

From the boundary condition in Eq. (52) it can be seen that $c' = 1$. Thus, for $\tau > X$,

$$u = \int X\exp(-\tau/2)I_1(1/2(\tau^2 - X^2)^{1/2})/(\tau^2 - X^2)^{1/2}\mathrm{d}\tau + \exp(-X/2) \tag{70}$$

It can be seen that the boundary conditions are satisfied by Eq. (70) and describes the transient temperature as a function of space and time that is governed by the hyperbolic wave diffusion and relaxation equation. The flux expression is given by Eq. (63). This can be integrated into a fluidized bed to surface heat transfer models as shown in Sharma and Turton [36].

V. Regimes of Heat Flux at an Interior Point inside the Medium

It can be seen that expressions for dimensionless heat flux and dimensionless temperature given by Eqs. (63) and (70) are valid only in the open interval for $\tau > X$. When $\tau = X$, the wave front condition results and the dimensionless heat flux and temperature are identical as

$$q^* = u = \exp(-X/2) = \exp(-\tau/2) \tag{71}$$

When $X > \tau$, the transformation variable can be redefined as $\eta = X^2 - \tau^2$. Equation (54) becomes

$$\eta^2\partial^2 W/\partial\eta^2 + \eta^2\partial W/\partial\eta + \eta W/16 = 0 \tag{72}$$

The general solution for this Bessel equation is given by

$$W = c_1 J_0(1/2\eta^{1/2}) + c_2 Y_0(1/2\eta^{1/2}) \tag{73}$$

The wave temperature W is finite when $\eta = 0$ and hence it can be seen that c_2 can be seen to be zero. The c_1 can be solved from the boundary condition given in Eq. (52). The expression in the open interval or the dimensionless heat flux for times τ, smaller than X is thus

$$q^* = c_1 \exp(-\tau/2) J_0[1/2(X^2 - \tau^2)^{1/2}] \tag{74}$$

On examining the Bessel function in Eq. (74) it can be seen that the first zero of the Bessel occurs when the argument becomes 2.4048. Beyond that point the Bessel function will take on negative values indicating a reversal of heat flux. There is no good reason for the heat to reverse in direction at short times. Hence Eq. (74) is valid from the wave front down to where the first zero of the Bessel function occurs. Thus the plane of zero transfer explains the initial condition verification from the solution.

By using the expression at the wave front for the dimensionless heat flux c_1 can be solved for and found to be 1. Equation. (74) can also be obtained directly from Eq. (63) by using $I_0(\eta) = J_0(i\eta)$. The expression for temperature in a similar vein for the open interval $X > \tau$ is thus

$$u = \int X \exp(-\tau/2) J_1(1/2(\tau^2 - X^2)^{1/2})/(\tau^2 - X^2)^{1/2} \mathrm{d}\tau + \exp(-X/2) \tag{75}$$

Consider a point X_p in the semi-infinite medium. Three regimes can be identified during the heating of this point from the surface as a function of time. This is illustrated in Fig. 4. The series expansion of the modified Bessel composite function of the first kind and zeroth order was used using a Microsoft Excel spreadsheet on a Pentium IV desktop microcomputer. The three regimes and the heat flux at the wave front are summarized as follows:

1) The first regime is a thermal inertia regime when there is no transfer.

2) The second regime is given by expression Eq. (74) for the heat flux and

$$q^* = \exp(-\tau/2) J_0(1/2(X^2 - \tau^2)^{1/2}) \tag{76}$$

The first zero of the zeroth order Bessel function of the first kind occurs at 2.4048. This is when

$$2.4048 = 1/2(X^2 - \tau^2)^{1/2} \quad \text{or} \quad \tau_{\text{lag}} = \sqrt{(X^2 - 23.132)} \tag{77}$$

Thus τ_{lag} is the thermal lag that will ensue before the heat flux is realized at an interior point in the semi-infinite medium at a dimensionless distance X from the surface. In Fig. 3 one value of X is used, that is, 5. Thus for points closer to the surface the time lag may be zero. Only for dimensionless distances greater than 4.8096, the time lag is finite. For distances *closer than* $4.8096\sqrt{(\alpha\tau_r)}$ the thermal lag experienced *will be zero*. For distances,

$$x > 4.8096\sqrt{(\alpha\tau_r)} \tag{78}$$

The time lag experienced is given by Eq. (77) and is $\sqrt{(X^2 - 4\beta_1^2)}$ where β_1 is the first zero of the Bessel function of the first kind and zeroth order and is 2.4048. In a similar fashion, the penetration distance of the disturbance for a considered instant in time, beyond which the change in initial temperature is zero, can be calculated as

$$X_{\text{pen}} = (23.132 + \tau_i^2)^{1/2}$$

3) The third regime starts at the wave front and is described by Eq. (76). This is shown in Fig. 4 for a point in the interior of the semi-infinite medium at a dimensionless distance 5 from the surface.

$$q^* = c_1 \exp(-\tau/2) I_0(1/2(\tau^2 - X^2)^{1/2}) \tag{79}$$

4) At the wave front, $q^* = u = \exp(-X/2) = \exp(-\tau/2)$.

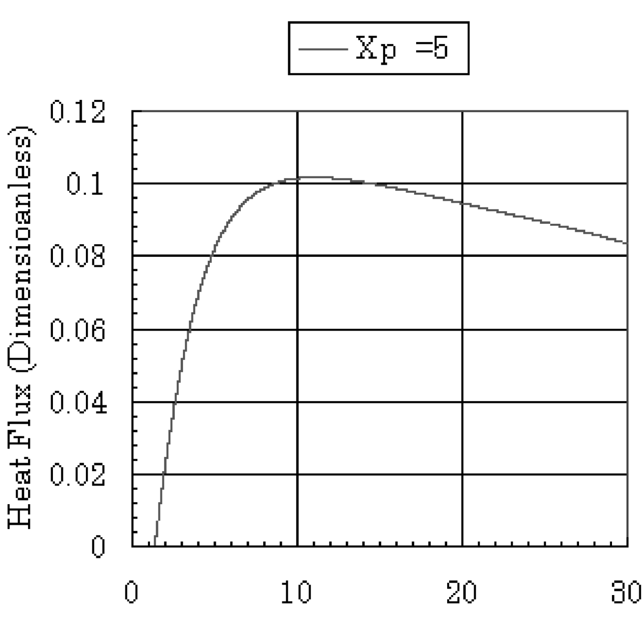

Fig. 4 Three regimes of heat flux in an interior point in a semi-infinite medium.

A. Approximate Solution

The expressions for transient temperature derived in the previous section need integration before use. More easily usable expressions can be developed by making suitable approximations. Realizing that for a PDE a set of functions instead of constants as in the case of ODE needs to be solved from the boundary conditions where c in Eq. (79) is allowed to vary with time. This results in an expression for transient temperature that is more readily available for direct use of the practitioner. Extensions to three dimensions in space are also straightforward in this method.

In this section, the exact solution for the constant wall temperature problem in semi-infinite medium in one dimension is revisited since the discussion by the method of Laplace transforms by Baumeister and Hamill. An expression that does not need further integration is attempted to be derived in this section. Consider a semi-infinite slab at initial temperature T_0, subjected to sudden change in temperature at one of the ends to T_1. The heat propagative velocity is $Vh = \sqrt{(\alpha/\tau_r)}$. The initial condition

$$t = 0, \qquad Vx, \qquad T = T_0 \tag{80}$$

$$T > 0, \qquad x = 0, \qquad T = T_1 \tag{81}$$

$$T > 0, \qquad x = \infty, \qquad T = T_0 \tag{82}$$

Obtaining the dimensionless variables;

$$u = (T - T_0)/(T_1 - T_0), \qquad \tau = t/\tau_r, \qquad X = x/\sqrt{(\alpha\tau_r)} \tag{83}$$

The energy balance on a thin spherical shell at x with thickness Δx is written. The governing equation [Eq. (15)] can be obtained after eliminating q between the energy balance equation and the derivative with respect to x of the flux equation and introducing the dimensionless variables. Suppose $u = \exp(-n\tau)w(X, \tau)$. By choosing $n = 1/2$, the damping component of the equation is removed. Thus for $n = 1/2$, Eq. (15) becomes

$$-w/4 + \partial^2 w/\partial\tau^2 = \partial^2 w/\partial X^2 \tag{84}$$

By inspecting the flux solution obtained by the method of Laplace transforms (Sharma [10]), consider the transformation variable η as

$$\eta = \tau^2 - X^2, \quad \text{for } \tau > X \tag{85}$$

$$\partial w/\partial\tau = (\partial w/\partial\eta)2\tau \tag{86}$$

$$\partial^2 w/\partial\tau^2 = (\partial^2 w/\partial\eta^2)4\tau^2 + 2(\partial w/\partial\eta) \tag{87}$$

In a similar fashion,

$$\partial^2 w/\partial X^2 = (\partial^2 w/\partial \eta^2)4X^2 + 2(\partial w/\partial h) \tag{88}$$

Substituting Eqs. (87) and (88) into Eq. (84)

$$(\partial^2 w/\partial \eta^2)4(\tau^2 - X^2) + 4(\partial w/\partial \eta) - w/4 = 0 \tag{89}$$

$$\eta^2 \partial^2 w/\partial \eta^2 + \eta \partial w/\partial \eta - \eta w/16 = 0 \tag{90}$$

Equation (90) can be recognized as the modified Bessel equation:

$$w = c_1 I_0[\sqrt{\eta}/2] + c_2 K_0[\sqrt{\eta}/2] \tag{91}$$

For $X = 0$, u is 1 or finite and hence it can be seen that w is also finite and hence $C_2 = 0$. Writing the expression for u,

$$u = c_1 \exp(-\tau/2)\{I_0[\sqrt{\eta}/2]\} \tag{92}$$

From the boundary condition (BC), Eq. (92) becomes

$$1 = c_1 \exp(-\tau/2) I_0(\tau/2) \tag{93}$$

c_1 can be eliminated by dividing Eq. (92) by Eq. (93) to yield

$$u = \left\{I_0\left[\sqrt{(\tau^2 - X^2)}\right]\Big/2\right\}/[I_0(\tau/2)] \tag{94}$$

for $\tau > X$. For $\tau < X$ or $t < x/v_h$

$$u = \left\{J_0\left[\sqrt{(X^2 - \tau^2)}\right]\Big/2\right\}/[I_0(\tau/2)] \tag{95}$$

Equations (94) and (95) are shown in Fig. 4 for dimensionless heat flux in a semi-infinite slab maintained at constant wall temperature at one of its edges. It can be inferred that an expression in time is used for c_1. A domain restricted solution for short and long times may be in order. For long times, $I_0(\tau/2)$ approximates as $\exp(\tau/2)/\sqrt{(2\pi\tau)}$.

$$1 = c_1 \exp(-\tau/2)[I_0(\tau/2)] = c_1 \exp(-\tau/2)\exp(\tau/2)/\sqrt{(2\pi\tau)} \tag{96}$$

$$c_1 = \sqrt{(2\pi\tau)} \tag{97}$$

or

$$u = \sqrt{(2\pi\tau)}\exp(-\tau/2)\left\{I_0\left[\sqrt{(\tau^2 - X^2)}\right]\Big/2\right\} \tag{98}$$

Thus the temperature solution in a semi-infinite medium when subject to a step change in temperature at one of the ends is obtained as an open interval solution. The dimensionless flux as earlier derived is

$$q = q''/\sqrt{(kC_p\rho/\tau_r)}(T_1 - T_0) = \exp(-\tau/2)\left\{I_0\left[1/2\sqrt{(\tau^2 - X^2)}\right]\right\} \tag{99}$$

Further the instantaneous surface flux is when $X = 0$,

$$q''/\sqrt{(kC_p\rho/\tau_r)}(T_1 - T_0) = \exp(-\tau/2) I_0(\tau/2) \tag{100}$$

B. Extension to Three Dimensions

Equation (15) in three dimensions can be written as

$$\partial u/\partial \tau + \partial^2 u/\partial \tau^2 = \partial^2 u/\partial X^2 + \partial^2 u/\partial Y^2 + \partial^2 u/\partial Z^2 \tag{101}$$

After dividing by $\exp(-\tau/2)$, the equation becomes

$$-w/4 + \partial^2 w/\partial \tau^2 = \partial^2 w/\partial X^2 + \partial^2 w/\partial Y^2 + \partial^2 w/\partial Z^2 \tag{102}$$

Let

$$\eta = \tau^2 - X^2 - Y^2 - Z^2 \tag{103}$$

$$(\partial^2 w/\partial \eta^2)4(\tau^2 - X^2 - Y^2 - Z^2) + 8(\partial w/\partial h) - w/4 = 0 \tag{104}$$

$$\eta^2 \partial^2 w/\partial \eta^2 + 2\eta \partial w/\partial \eta - \eta w/16 = 0 \tag{105}$$

Comparing Eq. (105) with the generalized Bessel equation;

$$b = 0, \quad a = 2, \quad c = 0, \quad s = 1/2, \quad d = -1/16$$

$$p = 2\sqrt{(1/4)} = 1 \text{ (order)} \tag{106}$$

$$w = c_{1/}\eta^{1/2} I_1[\sqrt{(\eta)}/2] + c_{2/}\eta^{1/2} K_1[\sqrt{(\eta)}/2]$$

It can be deduced that c_2 is zero as w is finite when $\eta = 0$. The boundary condition as a point temperature at the origin gives rise to $\tau > \sqrt{(X^2 + Y^2 + Z^2)}$:

$$u = \left[\tau/\sqrt{(\tau^2 - X^2 - Y^2 - Z^2)}\right] I_1\left[\sqrt{(\tau^2 - X^2 - Y^2 - Z^2)}/2\right]\Big/ I_1(\tau/2) \tag{107}$$

For $\sqrt{(X^2 + Y^2 + Z^2)} > \tau$, $\tau > 0$

$$u = \left[\tau/\sqrt{(X^2 + Y^2 + Z^2 - \tau^2)}\right] J_1\left[\sqrt{(X^2 + Y^2 + Z^2 - \tau^2)}/2\right]\Big/ I_1(\tau/2) \tag{108}$$

C. Periodic Boundary Condition

The storage coefficient S with the units of $W/m^3/K$ and given by $S = (\rho C_p/\tau_r)$ can be a critical parameter in the design of substrates of high speed processors in a similar vein to the thermal conductivity k (W/m/K) and heat transfer coefficient ($W/m^2/K$). The ratio of the thermal mass to the relaxation time of the material may be an indicator of the heat stored in the material. Especially during periodic phenomena this may become an important consideration. With the increase in speed of the microprocessors the periodic heating of the surface of a substrate becomes of interest. Given the time scales of the period of the microprocessor and some of the short time scale anomalies of Fourier heat conduction it is of interest to evaluate other expressions other than Fourier such as the Cattaneo and Vernotte non-Fourier heat conduction and relaxation equation, dual phase lag model subject to a periodic boundary condition.

Consider a semi-infinite slab at initial temperature T_0, imposed by a periodic temperature at one of the ends by $T_0 + T_1\cos[wt)$. The transient temperature as a function of time and space in one dimension is obtained. Obtaining the dimensionless variables:

$$u = (T - T_0)/(T_1), \quad \tau = t/\tau_r, \quad X = x/\sqrt{(\alpha\tau_r)} \tag{109}$$

The energy balance on a thin shell at x with thickness Δx is written. The governing equation [Eq. (15)] is obtained after eliminating q between the energy balance equation and the derivative with respect to x of the flux equation and introducing the dimensionless variables. The initial condition is

$$t = 0, \quad T = T_0, \quad u = 0 \tag{110}$$

The boundary conditions are

$$X = \infty, \quad T = T_0, \quad u = 0 \tag{111}$$

$$X = 0, \quad T = T_0 + T_1\cos(\omega t), \quad u = \cos(\omega^* t) \tag{112}$$

Let us suppose that the solution for u is of the form $f(x)\exp(-i\omega^* t)$ for $\tau > 0$, where ω is the frequency of the temperature wave imposed on the surface and T_1 is the amplitude of the wave. Equation (15) becomes

$$(-i\omega^*)f\exp(-i\omega^*\tau) + (i^2\omega^{*2})f\exp(-i\omega\tau) = f''\exp(-i\omega^*\tau) \tag{113}$$

$$i^2 f(\omega^{*2} + i\omega^*) = f'' \tag{114}$$

$$f(X) = c\exp[-iX\omega^*\sqrt{(\omega^* + i)}] \tag{115}$$

d can be seen to be zero as at $X = \propto$, $u = 0$:

$$u = c\exp[-iX\omega^* \sqrt{(\omega^* + i)}]\exp(-i\omega^* t) \quad (116)$$

From the boundary condition at $X = 0$,

$$\cos(\omega^* t) = \mathrm{Re}[c\exp(-i\omega^* t)] \quad \text{or} \quad c = 1 \quad (117)$$

$$\begin{aligned} u &= \exp(-X\omega^*(A + iB)\exp(-i\omega^*\tau) \\ &= \exp(-A\omega^* X)\exp[-i(BX\omega^* + \omega^*\tau)] \end{aligned} \quad (118)$$

where

$$A + iB = i\sqrt{(\omega^* + i)} \quad (119)$$

Squaring both sides

$$A^2 - B^2 + 2ABi = i^2(\omega^* + i) = -\omega^* - i \quad (120)$$

$$A^2 - B^2 = -\omega^*, \qquad 2AB = -1 \quad \text{or} \quad B = -1/2A \quad (121)$$

or

$$A^2 - 1/4A^2 = -\omega^* \quad (122)$$

$$A^2 = \left[-\omega^* \pm \sqrt{(\omega^{*^2} + 1)}\right]/2, \qquad B = -1/2A \quad (123)$$

Obtaining the real part,

$$(T - T_0)/(T_1) = u = \exp(-A\omega^* X)\cos[\omega^*(BX + \tau)] \quad (124)$$

The time lag in the propagation of the periodic disturbance at the surface is captured by the above relation. Thus the boundary conditions can be seen to be satisfied by Eq. (124). In a similar vein to the supposition of $f(x)\exp(-i\omega^*\tau)$ the heat flux q can be supposed to be of the form $q^* = g(x)\exp(-i\omega^*\tau)$. From Eq. (124) and the suppositions of u and q we have

$$g = f'/(1 - i\omega^*) \quad (125)$$

Combining the f from Eq. (115) into Eq. (125)

$$q^* = -\omega^*(A + iB)\exp[-X\omega^*(A + iB)]\exp(-i\omega^* t) \quad (126)$$

$$= -\omega^*(A + iB)\exp(-A\omega^* X)\exp[-i(BX\omega^* + \omega^*\tau)] \quad (127)$$

$$\begin{aligned} &= -\omega^*(A + iB)\exp(-A\omega^* X)[\cos(BX\omega^* + \omega^*\tau) \\ &\quad + i\sin(BX\omega^* + \omega^*\tau)] \end{aligned} \quad (128)$$

Obtaining the real part

$$\begin{aligned} q &= \sqrt{(kS)}\omega^*\exp(-A\omega^* X)[B\sin(\omega^*(BX + \tau)] \\ &\quad - A\cos[\omega^*(BX + \tau)] \end{aligned} \quad (129)$$

where $q^* = q/\sqrt{(kS)}$, k is the thermal conductivity, and S the storage coefficient $(\rho C_p/\tau_r)$. Thus the sustained part of the solution is periodic with a time lag from the periodic boundary condition imposed on the surface. The heat flux is determined by the negative temperature gradient and the accumulation term in the Cattaneo and Vernotte equation. For certain values it can be seen that the heat flux can reverse in direction. (Fig. 5). The flux reversal can be an issue in substrate design of high speed processors.

VI. Conclusions

The damped wave conduction and relaxation equation was derived from the free electron theory. It can also be derived from the kinetic theory of gases [10] and the viscoelastic spring and dashpot series model. The acceleration regime of the electron from the steady rest velocity to the steady drift velocity is neglected. When this is neglected Fourier's law of heat conduction can be derived. When the acceleration of the free electron on account of an imposed driving force such as the temperature gradient is taken into account, the Cattaneo and Vernotte equation results. The electron is believed by Lorentz and Drude to collide with the obstacle and lose the drift velocity and collapse to the rest velocity. Again it begins to accelerate and so on and so forth. The relaxation time in the damped wave conduction and relaxation equation is a third of the collision time of the electron and the material. Free electron theory has been widely accepted in explaining the observations in the theory of electric conduction. By Lorenz analogy the ratio of the electrical conductivity and thermal conductivity and temperature was found to be universal constant. Even for metals, the relaxation time can be a non-negligible parameter. The recent work of Mitra et al. [22] and Kaminski [23] has shown that for materials with a nonhomogeneous inner structure the relaxation time can be measured to 10–15 s. The relaxation time can be related to the thermal diffusivity by rearranging the terms obtained from the free electron theory. The electron density can be converted to a density and the Boltzmann constant to a heat capacity term for ideal gases to obtain an expression for thermal diffusivity. With the advent of personal computers and continued interest in transient phenomena the damped wave conduction and relaxation can be investigated further.

The temperature solution for a finite slab subject to constant wall temperature using the damped wave conduction and relaxation equation presented by earlier investigators showed that there can be under certain conditions a temperature overshoot and a violation of second law of thermodynamics. In this study the second order hyperbolic PDE in two variables was solved by two methods. In the first method, the damping term was first removed from the equation by a $u = \exp(-n\tau)$ substitution. The resulting equation in wave temperature was solved by the method of separation of variables. Three of the four conditions in space and time used by the earlier investigators were retained in this study. These were the boundary temperature, by symmetry the derivative of temperature with respect to space becoming zero at $x = 0$, and the initial temperature of the slab. The fourth time condition was the steady state condition in place of the derivative of the temperature with respect to time in the initial condition. At steady state the wave temperature $W = u\exp(\tau/2)$ becomes a product of zero times infinity. As a result in the time domain solution of the wave temperature the term that grows exponentially with time is neglected leaving behind the term that exponentially decays with time. The resulting solution is well bounded and within the second law of thermodynamics. No temperature overshoot can be seen in this solution. For small slabs, $a < \pi (\alpha\tau_r)^{1/2}$, the temperature exhibits subcritical damped oscillations. This is shown in Fig. 2 for the centerline temperature of the slab. A parametric study on the relaxation time parameter was completed and the results presented in Fig. 3. The subcritical damped oscillations can be seen for large relaxation time and when the relaxation time is small the centerline temperature exhibits a monotonic decay. Compared with some of the analytical solution given in the literature for a semi-infinite slab (Baumeister and Hamill [30]) the solution is for the complete domain from the initial condition to steady state in time. The solution presented in the literature for the semi-infinite slab showed a discontinuity at the wave front. A time taken to steady state can be read off from the graph for small slabs.

This is a new feature in transient heat transfer that arises from the solution of the hyperbolic PDE. It appears as though there is another mode of heat transfer in addition to the molecular conduction, convection, and radiation. This can be the wave conduction. This can be from the accumulation at the surface effects at a molecular level or the acceleration regime in the free electron theory. When attempting to empirically fit two phenomena, Fourier's law of heat conduction deviates from reality for some circumstances that are important in industrial practice. Thus the circumstances which determine whether the transfer of heat is direct exponential decay or cosinuous can be derived theoretically using the damped wave conduction and relaxation equation. The Fourier infinite series solution has been found to be well bounded for the surface flux condition where there was found a blowup in the parabolic solution.

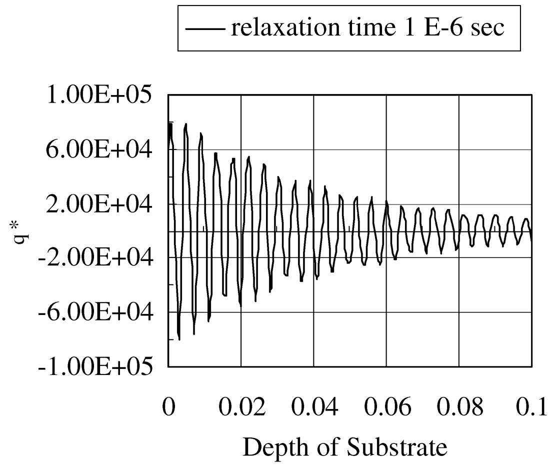

Fig. 5 Dimensionless heat flux with depth of the substrate $\tau = 10$; $\tau_r = 10^{-6}$ s; damped wave conduction for the periodic boundary condition.

An aliter was presented where the direct solution of the hyperbolic PDE was undertaken. A transformation $\eta = X^2 - \tau^2$ can lump the ballistic accumulation term and the Fourier term in the damped wave conduction and relaxation equation to two terms in the similarity variable. The resulting PDE in the domain of $X > \tau$ becomes elliptic from hyperbolic. This equation can be solved for the method of separation of variables. The solution is an infinite Fourier–Bessel series in the composite spatiotemporal variable. When separating the two functions after differentiation one is a function in time only and the other is a function of the spatiotemporal variable η. Both will be equal when the expression containing the spatiotemporal variable becomes a function of time only and this can be set equal to f. The solution is seen to obey the time and space conditions. The eigenvalues were solved for the boundary condition at the surface and from the roots of the Bessel function of the zeroth order and first kind.

Both solutions are well bounded without any temperature overshoot and disobeyance of Clausius inequality.

The analytical solution presented by Baumeister and Hamill [30] showed a discontinuity at the wave front. This has been improved upon by a substitution, $\eta = \tau^2 - X^2$. The solution is in the form of a Bessel composite function of the spatiotemporal variable and exhibits three different regimes: an inertial regime of lag and zero transfer, a rising regime described by a Bessel composite spatiotemporal function, and a third regime of a modified Bessel composite function of the spatiotemporal variable and a falling regime in the expression for heat flux. The three regimes exhibited by the dimensionless shear stress are shown in Fig. 4. Readily usable solutions are obtained for the dimensionless heat flux and temperature. Some space time symmetry is seen in the solution. The wave front solution can be obtained directly from the transformed Bessel differential equation. The manifestation of the relaxation time during a periodic boundary condition is shown in Fig. 5 upon obtaining the solution using the method of complex temperature. It can be seen it is an overdamped system. A storage coefficient is defined which plays a critical role in the oscillatory regimes and is given by $S = \rho C_p/\tau_r$ and has the units of $W/m^3/K$.

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